A Reflection Principle for Three Vicious Walkers

William Y. C. Chen[1], Donna Q. J. Dou[2], Terence Y. J. Zhang[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]qjdou@cfc.nankai.edu.cn, [3]terry@mail.nankai.edu.cn

Abstract

We establish a reflection principle for three lattice walkers and use this principle to reduce the enumeration of the configurations of three vicious walkers to that of configurations of two vicious walkers. In the combinatorial treatment of two vicious walkers, we make connections to two-chain watermelons and to the classical ballot problem. Precisely, the reflection principle leads to a bijection between three walks (L_1, L_2, L_3) such that L_2 intersects both L_1 and L_3 and three walks (L_1, L_2, L_3) such that L_1 intersects L_3. Hence we find a combinatorial interpretation of the formula for the generating function for the number of configurations of three vicious walkers, originally derived by Bousquet-Mélou by using the kernel method, and independently by Gessel by using tableaux and symmetric functions.

Keywords: vicious walkers, watermelon, Catalan numbers, Ballot numbers, reflection principle.

AMS Classification Numbers: 82B23; 05A15

1 Introduction

The vicious walker model was introduced by Fisher [5] in 1984 and has drawn much attention. A walker is said to be vicious if he does not like to meet any other walker at any point. Formally speaking, a configuration of r vicious walkers, called r vicious walks, of length n, is an r-tuple of pairwise nonintersecting lattice walks of length n, consisting of up steps U (i.e., $(1,1)$) and down steps D (i.e., $(1,-1)$), starting from $(0,2i_1),(0,2i_2),\ldots,(0,2i_r)$ and ending at $(n,e_1),(n,e_2),\ldots,(n,e_r)$ where $i_r > \cdots > i_2 > i_1 = 0$ and $e_r > \cdots > e_2 > e_1$. Precisely, two lattice paths are said to be nonintersecting if they do not share any common points. In particular, a watermelon of length n is a configuration consisting of r chains, or paths, of length n which start at the points $(0,0),(0,2),\ldots,(0,2r-2)$ and end at the points $(n,k),(n,k+2),\ldots,(n,k+2r-2)$ for some k. In other words, a watermelon

is a vicious walker configuration starting at adjacent points and ending at adjacent points. Note that two lattice points are said to be adjacent if they are on the same vertical line and their y-coordinates differ by 2. It is known that configurations of vicious walkers can be represented by tableaux. So the theory of symmetric functions can be employed to study vicious walkers, see [6, 8, 9, 10, 12, 13].

The main objective of this paper is to present a combinatorial approach to the enumeration of configurations of three vicious walkers. Let us fix the starting points $(0,0)$, $(0,2i)$ and $(0,2i+2j)$. Let $V(i,j,n)$ be the set of three vicious walks (L_1, L_2, L_3) of length n, where L_1 is the path of the first walker starting from $(0,0)$, L_2 is the path of the second walker starting from $(0,2i)$, and L_3 is the path of the third walker starting from $(0,2i+2j)$. Define the generating function $V_{i,j}(t)$ to be

$$V_{i,j}(t) = \sum_{n=0}^{\infty} |V(i,j,n)| t^n, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality of a set.

The enumeration of configurations of three vicious walkers has been solved independently by Bousquet-Mélou [1] by using the kernel method, and by Gessel [7] by using tableaux and symmetric functions. They obtained a formula for $V_{i,j}(t)$ in terms of the generating function of the Catalan numbers.

Let $C(t)$ be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, that is,

$$C(t) = \sum_{n=0}^{\infty} C_n t^n.$$

Recall that $C(t)$ satisfies the recurrence relation

$$C(t) = 1 + tC^2(t). \tag{1.2}$$

Let

$$D(t) = tC^2(t) = C(t) - 1 = \sum_{n=0} C_{n+1} t^{n+1}. \tag{1.3}$$

The following elegant formula is due to Bousquet-Mélou [1] and Gessel [7].

Theorem 1.1 (Bousquet-Mélou [1] and Gessel [7])

$$V_{i,j}(t) = \frac{1}{1-8t}(1 - D^i(2t))(1 - D^j(2t)). \tag{1.4}$$

In view of the relation (1.3) and the identity

$$\left(\frac{1+D(t)}{1-D(t)}\right)^2 = \frac{1}{1-4t}, \tag{1.5}$$

Gessel derived the following form of the formula for $V_{i,j}(t)$.

Theorem 1.2 (Gessel [7]) *For any $i, j \geq 1$, we have*

$$V_{i,j}(t) = C^2(2t)\big(1+D(2t)+\cdots+D^{i-1}(2t)\big)\big(1+D(2t)+\cdots+D^{j-1}(2t)\big). \tag{1.6}$$

Both Bousquet-Mélou [1] and Gessel [7] proposed the problem of finding a combinatorial interpretation of the formula for $V_{i,j}(t)$. The question of Bousquet-Mélou is concerned with the formula (1.4), while the question of Gessel is concerned with the formula in the form of (1.6). In this paper, we will present a combinatorial interpretation of (1.4). As will be seen, the algebraic manipulations to transform the formula (1.4) to (1.6) can be explained combinatorially. So we have obtained combinatorial interpretations of both formulas (1.4) and (1.6).

We also take a different approach to the enumeration of configurations of two vicious walkers. By reformulating the problem in terms of pairs of intersecting walks, we give a decomposition of a pair of converging walks, that is, two walks that do not intersect until they reach the same ending point, into two-chain watermelons, or 2-watermelons. Then we can use Labelle's formula for the number of 2-watermelons of length n to derive the formula for the number of two vicious walks of length n. In the last section, we make a connection between pairs of converging walks and the classical ballot numbers, by applying the Labelle merging algorithm, in the form presented by Chen, Pang, Qu and Stanley [3],

2 The Reflection Principle

In this section, we will establish a reflection principle so that we can reduce the enumeration of three vicious walkers to that of two vicious walkers. This reduction leads to a combinatorial interpretation of the formula for $V_{i,j}(t)$, as defined by (1.1).

Let us recall some basic definitions. Two walks L_1 and L_2 are said to be intersecting, denoted $L_1 \cap L_2 \neq \emptyset$, if L_1 and L_2 share a common point. Let $U(i,j,n)$ be the set of all 3-walks (L_1, L_2, L_3) of length n, where L_1, L_2 and L_3 start from $(0,0)$, $(0,2i)$ and $(0,2i+2j)$ respectively. Let

$$U_{i,j}(t) = \sum_{n=0}^{\infty} |U(i,j,n)| t^n.$$

It is obvious that

$$U_{i,j}(t) = \frac{1}{1-8t}. \tag{2.1}$$

We use $W_{12}(n)$, or W_{12} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 and L_2 are nonintersecting. Similarly, we use $W_{23}(n)$, or W_{23} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_2 and L_3 are nonintersecting. Clearly, the set $V(i,j,n)$ of three vicious walks of length n can be expressed as $W_{12} \cap W_{23}$. By the principle of inclusion and exclusion, we see that

$$|V(i,j,n)| = |W_{12} \cap W_{23}| = |W_{12}| + |W_{23}| - |W_{12} \cup W_{23}|. \tag{2.2}$$

In order to compute $|W_{12} \cup W_{23}|$, we let $M_{12,23}(n)$, or $M_{12,23}$ for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_2 intersects both L_1 and L_3. Clearly, we have

$$|W_{12} \cup W_{23}| = |U(i,j,n)| - |M_{12,23}|. \tag{2.3}$$

We are now in a position to establish a reflection principle to deal with the enumeration of $M_{12,23}(n)$. Let $M_{13}(n)$, or M_{13} for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 intersects L_3. Then we have the following correspondence.

Theorem 2.1 *For $n \geq 1$, there exists a bijection between $M_{12,23}(n)$ and $M_{13}(n)$.*

Proof. We construct a map Φ from $M_{12,23}(n)$ to $M_{13}(n)$ as follows. Let (L_1, L_2, L_3) be a 3-walk in $M_{12,23}(n)$. We consider the following two cases. If $L_1 \cap L_3 \neq \emptyset$, then it is clear that $(L_1, L_2, L_3) \in M_{13}(n)$. In this case, we define $\Phi((L_1, L_2, L_3)) = (L_1, L_2, L_3)$.

We may now assume that $L_1 \cap L_3 = \emptyset$. We first consider the case that L_2 meets L_1 before it meets L_3. Suppose that P is the first intersection point of L_2 and L_1. We now conduct the usual reflection operation on L_1 and L_2, and denote the resulting paths by L_1' and L_2'. Namely, L_1' consists of the first segment of L_1 up to the point P followed by the last segment of L_2 starting from the point P, and L_2' consists of the first segment of L_2 up to the point P followed by the last segment of L_1 starting from the point P. Figure 2.1 is an illustration of the reflection.

Let $L_3' = L_3$ and $\Phi((L_1, L_2, L_3)) = (L_1', L_2', L_3')$. It is clear that L_1' must meet L_3'. Thus we have $(L_1', L_2', L_3') \in M_{13}(n)$.

It is not difficult to see that the above procedure is reversible. We are still left with the case when L_2 intersects L_3 before meeting L_1. This case

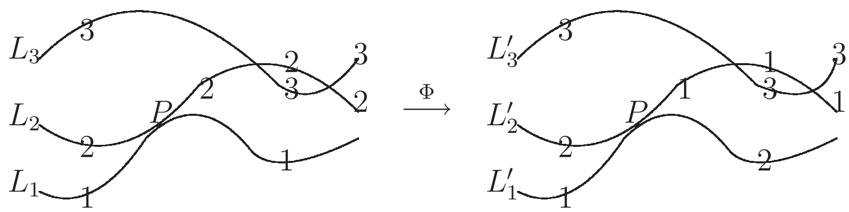

Figure 2.1: The reflection principle.

is analogous to the case that we have considered. Thus we have reached the conclusion that Φ is a bijection. ∎

Combining (2.2), (2.3) and Theorem 2.1, we obtain the following relation

$$|V(i,j,n)| = |W_{12}| + |W_{23}| + |M_{13}| - |U(i,j,n)|. \tag{2.4}$$

Let W_{13} be the set of three walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 never meets L_3, and define the generating functions for $|W_{12}|$, $|W_{23}|$ and $|W_{13}|$ by $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ respectively. From (2.4) it follows that

$$|V(i,j,n)| = |W_{12}| + |W_{23}| - |W_{13}|. \tag{2.5}$$

Proposition 2.2

$$V_{i,j}(t) = W_{12}(t) + W_{23}(t) - W_{13}(t). \tag{2.6}$$

The above formula can be viewed as a reduction of the three vicious walkers problem to that of two vicious walkers. Let $N(i,n)$ be the set of two vicious walks (L_1, L_2) of length n starting at $(0,0)$ and $(0,2i)$ respectively, and denote the corresponding generating function by

$$N_i(t) = \sum_{n=0}^{\infty} |N(i,n)| t^n.$$

Bousquet-Mélou [1] and Gessel [7] obtained the following formula

$$N_i(t) = \frac{1}{1-4t}(1 - D^i(t)). \tag{2.7}$$

As pointed out by Gessel [7], the above formula for $N_i(2t)$ can be deduced from the formula (1.6) for $V_{i,j}(t)$ by taking the limit $j \to \infty$, and by using the identity (1.5).

Using the above formula for $N_i(t)$, one can derive the following formulas for the generating functions $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$:

$$W_{12}(t) = \frac{1 - D^i(2t)}{1 - 8t}, \; W_{23}(t) = \frac{1 - D^j(2t)}{1 - 8t}, \; W_{13}(t) = \frac{1 - D^{i+j}(2t)}{1 - 8t}. \quad (2.8)$$

Clearly, formula (1.4) in Theorem 1.1 follows from the above formulas and the relation (2.6).

We note that Gessel [7] obtained the following identity

$$V_{i,j}(t) = N_i(2t) + N_j(2t) - N_{i+j}(2t), \quad (2.9)$$

in accordance with the combinatorial statement (2.6) derived from the reflection principle.

As to the question of finding a combinatorial interpretation of the generating function formula (1.4), the reflection principle (Theorem 2.1) along with the combinatorial interpretations of the formulas for $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ can be considered as an answer because the principle of inclusion and exclusion for two sets can be easily justified combinatorially. In the next section, we will present a combinatorial treatment of the formula (2.7) for two vicious walkers. Moreover, we note that one can give a combinatorial reasoning of the transformation from the formula (1.4) to the formula (1.6).

It is to deduce (1.6) from (1.4) by utilizing the identity (1.5), which can be explained combinatorially in two steps. The first step is to show that

$$4^n = \sum_{k=0}^{2n} \binom{2k}{k} \binom{2n-2k}{n-k}, \quad (2.10)$$

which is equivalent to the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}. \quad (2.11)$$

There are several combinatorial proofs of (2.10), see, for example, Kleitman [11] and Marta [15]. The second step is to show that

$$\frac{1 + D(t)}{1 - D(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n. \quad (2.12)$$

Note that $\frac{1+D(t)}{1-D(t)}$ can be written as $\frac{C(t)}{1-tC^2(t)}$. A combinatorial interpretation of the identity

$$\frac{C(t)}{1 - tC^2(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n$$

is given by Chen, Li and Shapiro [2] in terms of doubly rooted plane trees and the butterfly decomposition.

3 Converging Walks and 2-Watermelons

In this section, we present a different approach to the two vicious walkers problem by counting pairs of converging walks. A pair of walks is said to be converging if they never meet until they reach a common ending point. We will show that pairs of converging walks can be enumerated by applying Labelle's formula for two-chain watermelons, or 2-watermelons [14]. Precisely, we will give a decomposition of a pair of converging walks into 2-watermelons.

Recall that $M_{13}(n)$ is defined in the previous section. Let $M_{12}(n)$, or M_{12} for short, be the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 intersects L_2. Similarly, we can define $M_{23}(n)$, or M_{23} for short. Clearly, we have

$$|M_{12}| = |U(i,j,n)| - |W_{12}|, \quad |M_{23}| = |U(i,j,n)| - |W_{23}|.$$

From (2.4) it follows that

$$|V(i,j,n)| = |U(i,j,n)| + |M_{13}| - |M_{12}| - |M_{23}|.$$

Let $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ denote the generating functions for $|M_{12}(n)|$, $|M_{23}(n)|$ and $|M_{13}(n)|$, respectively.

Proposition 3.1 *We have*

$$V_{i,j}(t) = U_{i,j}(t) + M_{13}(t) - M_{12}(t) - M_{23}(t). \tag{3.1}$$

We will show that $M_{12}(t)$, $M_{13}(t)$ and $M_{23}(t)$ can be computed by using Labelle's formula for 2-watermelons.

Proposition 3.2 (Labelle [14]) *The number of* 2*-watermelons with each walk having* n *steps is* C_{n+1}.

By Labelle's formula, one sees that the generating function of the number of 2-watermelons equals $C^2(t)$. Note that 2-watermelons of length n correspond to pairs of converging walks of length $n+1$ with adjacent starting points. In general, let $T(i,n)$ be the set of pairs of converging walks (L_1, L_2) of length n, where L_1 starts from $(0,0)$ and L_2 starts from $(0,2i)$. Define

$$T_i(t) = \sum_{n\geq 0} |T(i,n)| t^n.$$

Proposition 3.3 *For any* $i \geq 1$, $T_i(t) = D^i(t)$.

Proof. Let $L_1 = A_0A_1 \dots A_n$ and $L_2 = B_0B_1 \dots B_n$, where a walk is represented by a sequence of points. For $0 \le k \le i$, let j_k be the minimum index such that the difference of the y-coordinates of (A_{j_k}, B_{j_k}) equals to $2i - 2k$. It is clear that $j_0 = 0$ and $j_i = n$. We now decompose (L_1, L_2) into i 2-walks: $(L_1^{(1)}, L_2^{(1)}), \dots, (L_1^{(i)}, L_2^{(i)})$, where $L_1^{(k)} = A_{j_{k-1}}A_{j_{k-1}+1} \dots A_{j_k}$ and $L_2^{(k)} = B_{j_{k-1}}B_{j_{k-1}+1} \dots B_{j_k}$. Figure 3.1 is an illustration of the decomposition.

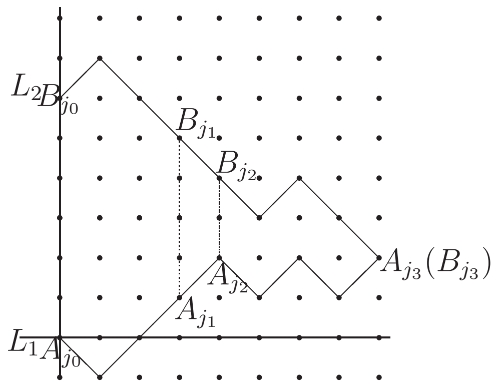

Figure 3.1: The decomposition of a pair of converging walks.

Observe that by the choice of j_k, the rightmost pair of steps in $(L_1^{(k)}, L_2^{(k)})$ must be (U, D). Moreover, if we delete this pair of steps, the resulting upper walk can be lowered $2i - 2k$ units without intersecting the lower walk to form a 2-watermelon. See Figure 3.2 for an example.

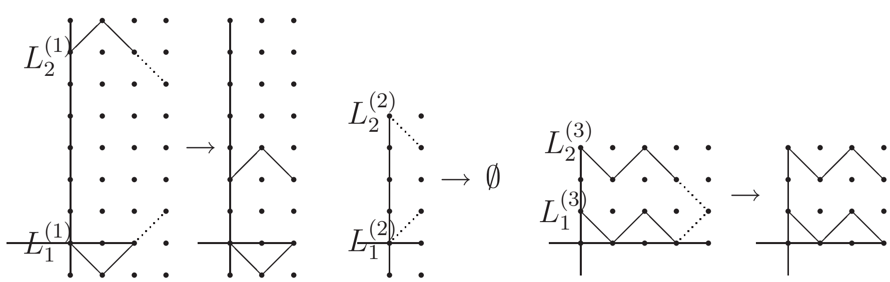

Figure 3.2: From 2-walks to 2-watermelons.

By Proposition 3.2, The generating function for the number of 2-walks $(L_1^{(k)}, L_2^{(k)})$ equals $D(t) = t \cdot C^2(t)$. This completes the proof. ∎

Let $M(i, n)$ be the set of intersecting 2-walks (L_1, L_2) of length n, where L_1 and L_2 start from $(0, 0), (0, 2i)$ respectively. Define

$$M_i(t) = \sum_{n \ge 0} |M(i, n)| t^n.$$

Observe that every pair of intersecting paths (L_1, L_2) can be decomposed into a pair of converging paths and a pair of arbitrary paths starting from the same point. Thus we have the following formula.

Corollary 3.4 *For any $i \geq 1$,*

$$M_i(t) = \frac{D^i(t)}{1-4t}.$$

It is obvious that

$$M_i(t) + N_i(t) = \frac{1}{1-4t}. \tag{3.2}$$

So the formula (2.7) for $N_i(t)$ can be deduced from the above formula. It is easy to see that $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ can be computed by using the above formula for $M_i(t)$. So we get

$$M_{12}(t) = \frac{D^i(2t)}{1-8t}, \quad M_{23}(t) = \frac{D^j(2t)}{1-8t}, \quad M_{13}(t) = \frac{D^{i+j}(2t)}{1-8t}, \tag{3.3}$$

in agreement with (2.8). Substituting (3.3) into (3.1), we obtain Theorem 1.1.

4 Connection to the Ballot Numbers

In this section, we put the Labelle merging algorithm in a more general setting, and show that the direct correspondence formulated by Chen, Pang, Qu and Stanley [3] leads to a connection between pairs of converging walks and the classical ballot numbers.

Let us recall the direct correspondence given in [3]. We will represent a walk as a sequence of steps rather than points. Let (L_1, L_2) be a 2-watermelon of length n, and let $L_1 = p_1 p_2 \cdots p_n$ and $L_2 = q_1 q_2 \cdots q_n$, where $p_i, q_i = U$ or D. Set $U' = D$ and $D' = U$. Using the direct correspondence in [3], the watermelon (L_1, L_2) can be represented by a Dyck path of length $2n+2$:

$$U q_1 p_1' q_2 p_2' \cdots q_n p_n' D.$$

It is not difficult to see that the above correspondence is a bijection. Figure 4.1 gives an illustration.

Using the same idea, we may encode a pair of converging walks (L_1, L_2) in $T(i, n)$ by a partial Dyck path P in the sense that the starting point of P is not necessarily the point $(0, 0)$. We should note that the common definition of a partial Dyck path is a lattice path starting from the origin $(0, 0)$ with up and down steps not going below the x-axis. Define $P(i, n)$ to be the set of all

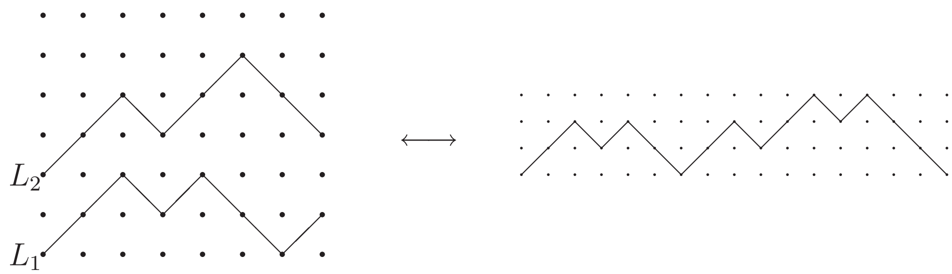

Figure 4.1: From a 2-watermelon to a Dyck path.

partial Dyck paths of length $2n$ which start from $(0, 2i)$ and never return to the x-axis except for the final destination. The following proposition establishes the connection between converging walks and partial Dyck paths.

Proposition 4.1 *For $n \geq 1$, there exists a bijection between $T(i,n)$ and $P(i,n)$.*

Proof. Given a pair of converging walks (L_1, L_2) in $T(i,n)$, let $L_1 = p_1 p_2 \cdots p_n$ and $L_2 = q_1 q_2 \cdots q_n$, where $p_i, q_i = U$ or D. Then (L_1, L_2) can be represented by a partial Dyck path P of length $2n$ starting from $(0, 2i)$:

$$P = q_1 p_1' q_2 p_2' \cdots q_n p_n'.$$

Clearly, P returns to the x-axis at the ending point and never touches the x-axis before the ending point, that is, $P \in P(i,n)$. It is easy to verify that the above correspondence is a bijection. Figure 4.2 is an illustration. ∎

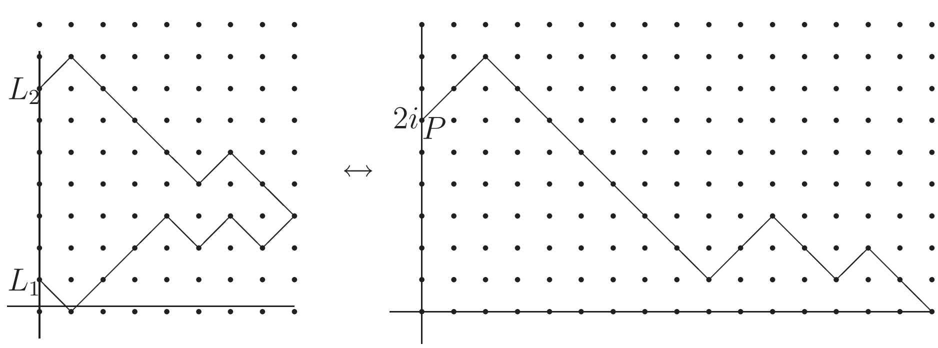

Figure 4.2: From a pair of converging walks to a partial Dyck path.

It is well known that the number of partial Dyck paths in $P(i,n)$ is given by the classical ballot number. Here we give a decomposition of a partial Dyck

path into Dyck paths in accordance with the generating function of $|T(i,n)|$ as given in Proposition 3.3.

Given a partial Dyck path P in $P(i,n)$, we can decompose P into i nonempty Dyck paths $P_1,\ldots,P_i$ via the following procedure. Let $P=A_0A_1\cdots A_{2n}$, where P is represented by the sequence of points rather than steps. Let $j_0=0$, and for $1\le k\le i$, let j_k be the minimum index such that the y-coordinate of A_{j_k} is two less than that of $A_{j_{k-1}}$. Then we can decompose P into i segments $Q_1,Q_2,\ldots,Q_i$, where Q_k is the segment of P starting at $A_{j_{k-1}}$ and ending at A_{j_k}. Observe that by the choice of j_k, the rightmost two steps of Q_k must be DD. Let P_k denote the Dyck path obtained from Q_k by deleting the last down step and adding an up step before the first step of Q_k. Evidently, P_k is a nonempty Dyck path. This completes the proof. ∎

To conclude this paper, we note that $|T(i,n)|$ can be computed by using the Lagrange inversion formula, or by using the formula for the number of Dyck paths of length $2n+2i$ with $2i$ returns to the x-axis, see Deutsch [4]. The explicit formula is as follows:

$$|T(i,n)|=\frac{i}{n}\binom{2n}{n-i}.$$

We also note that $|T(i,n)|$ can be expressed as the classical ballot number $b(n+i-1,n-i)$, where

$$b(n,i)=\binom{n+i}{i}-\binom{n+i}{i-1}=\frac{n+1-i}{n+1+i}\binom{n+i+1}{i},$$

see, for example, Riordan [16].

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